

# Particle creation by moving spherical shell in the dynamical Casimir effect

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## Abstract

The creation of massless scalar particles from the quantum vacuum by spherical shell with time varying radius is studied. In the general case of motion the equations are derived for the instantaneous basis expansion coefficients. The examples are considered when the mean number of particles can be explicitly evaluated in the adiabatic approximation.

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# 1 Introduction

The Casimir effect is one of the most interesting manifestations of nontrivial properties of the vacuum state in a quantum field theory (for reviews see [1, 2, 3, 4]) and can be viewed as a polarization of vacuum by boundary conditions. A new phenomenon, a quantum creation of particles (the dynamical Casimir effect) occurs when the geometry of the system varies in time. In two dimensional spacetime and for conformally invariant fields the problem with dynamical boundaries can be mapped to the corresponding static problem and hence allows a complete study (see [2, 4] and references therein). In higher dimensions the problem is much more complicated and is solved for some simple geometries. The vacuum stress induced by uniform acceleration of a perfectly reflecting plane is considered in [5]. The corresponding problem for a sphere expanding in the four-dimensional spacetime with constant acceleration is investigated by Frolov and Serebriany [6, 7] in the perfectly reflecting case and by Frolov and Singh [8] for semi-transparent boundaries. For more general cases of motion by vibrating cavities the problem of particle and energy creation is considered on the base of various perturbation methods [9, 10, 11, 12, 13, 14, 15, 16] (for more complete list of references see [16]). It have been shown that a gradual accumulation of small changes in the quantum state of the field could result in a significant observable effect. A new application of the dynamical Casimir effect has recently appeared in connection with the suggestion by Schwinger [17] that the photon production associated with changes in the quantum electrodynamic vacuum state arising from a collapsing dielectric bubble could be relevant for sonoluminescence (the phenomenon of light emission by a sound-driven gas bubble in a fluid [18]). For the further developments and discussions this quantum-vacuum approach see [21, 19, 20, 22, 23] and references therein. In the present paper we consider particle creation from the quantum scalar vacuum by expanding or contracting spherical shell with Dirichlet boundary conditions. In next section we derive the equations for the instantaneous basis expansion coefficients and the formula for the Bogoliubov coefficients. The examples when this coefficients can be explicitly found at adiabatic approximation are discussed in section 3.

## 2 General consideration

Consider a scalar field  $\varphi$  satisfying Dirichlet boundary condition on the surface of a sphere with time-dependent radius  $a = a(t)$ :

$$\left(\frac{\partial^2}{\partial t^2} - \Delta\right)\varphi(x, t) = 0, \quad \varphi|_{r=a(t)} = 0. \quad (1)$$

The corresponding eigenfunctions can be expanded in a series with respect to the instantaneous basis

$$\varphi_\alpha(x) = \sum_\beta q_{\alpha\beta}(t)\phi_\beta(x, a(t)), \quad (2)$$

where the collective index  $\alpha$  denotes the set of quantum numbers specifying the solution. Here  $\phi_\beta(x, a)\exp(-i\omega t)$  are the corresponding eigenfunctions for a static sphere with radius  $a$ :

$$\begin{aligned} \phi_\beta(x, a) &= \frac{\sqrt{2}j_l(j_{l,n}r/a)}{a^{3/2}j'_l(j_{l,n})}Y_{lm}(\theta, \varphi), \\ \beta &= (l, m, n), \quad l = 0, 1, 2, \dots, \quad -l \leq m \leq l, \quad n = 1, 2, \dots, \end{aligned} \quad (3)$$

with  $j_{l,n}$  being the  $n$ -th zero for the spherical Bessel function  $j_l(z)$ ,  $j_l(j_{l,n}) = 0$ ,  $Y_{lm}(\theta, \varphi)$  is the spherical harmonic. Note that the sets of quantum numbers are the same for static and dynamic cases. Function (2) satisfies boundary condition (1). Putting expression (2) into the field equation (1) we obtain

$$\sum_{\beta} \left\{ \phi_{\beta} [\ddot{q}_{\alpha\beta}(t) + \omega_{\beta}^2(t) q_{\alpha\beta}(t)] + 2\dot{q}_{\alpha\beta} \dot{\phi}_{\beta} + q_{\alpha\beta} \ddot{\phi}_{\beta} \right\} = 0, \quad \omega_{\beta}(t) = j_{l,n}/a(t), \quad (4)$$

where dot stands for the time-derivative. Let us multiply this equation by  $\phi_{\beta'}^*$  and integrate over the region inside a sphere at a given moment  $t$ . Using the orthonormality relation

$$\int \phi_{\beta} \phi_{\beta'}^* d^3x = \delta_{\beta\beta'}, \quad (5)$$

this yields

$$\ddot{q}_{\alpha\beta}(t) + \omega_{\beta}^2(t) q_{\alpha\beta}(t) = \sum_{\beta'} (2\dot{q}_{\alpha\beta'} g_{\beta'\beta} + q_{\alpha\beta'} g_{\beta'\beta}^{(1)}), \quad (6)$$

where we have introduced notations

$$g_{\beta'\beta} = - \int \dot{\phi}_{\beta'} \phi_{\beta}^* d^3x, \quad g_{\beta'\beta}^{(1)} = - \int \ddot{\phi}_{\beta'} \phi_{\beta}^* d^3x, \quad (7)$$

with integrations over the region inside the sphere. The relation between coefficients (7) can be found by making use the completeness condition for eigenfunction (3):

$$\sum_{\gamma} \phi_{\gamma}(\vec{r}) \phi_{\gamma}^*(\vec{r}') = \delta(\vec{r} - \vec{r}'). \quad (8)$$

This yields

$$g_{\beta'\beta}^{(1)} = \frac{\partial g_{\beta'\beta}}{\partial t} + \sum_{\gamma} g_{\beta'\gamma} g_{\beta\gamma}^*. \quad (9)$$

By taking into account this relation we note that the general structure of equations (6) for the instantaneous basis expansion coefficients is similar to that for the plane case. Using the standard orthonormality relations for the spherical harmonic it can be easily seen that

$$g_{\beta'\beta} = g_{n'n}^l \delta_{ll'} \delta_{mm'}, \quad g_{\beta'\beta}^{(1)} = g_{n'n}^{(1)l} \delta_{ll'} \delta_{mm'}. \quad (10)$$

To evaluate the integrals in (7) we use the formula for the integrals involving the Bessel spherical function

$$\int_0^1 z^2 j_l(az) j_l(bz) dz = \frac{a j_l'(a) j_l(b) - b j_l(a) j_l'(b)}{b^2 - a^2}. \quad (11)$$

From this formula the following expression for the coefficients can be obtained

$$g_{n'n}^l = h a_{n'n}^l, \quad g_{n'n}^{(1)l} = \dot{h} a_{n'n}^l + h^2 \sum_p a_{n'p}^l a_{np}^l, \quad h = \frac{\dot{a}}{a}, \quad (12)$$

where

$$\begin{aligned} a_{nn}^l &= 0, \\ a_{n'n}^l &= \frac{2j_{l,n'} j_{l,n}}{j_{l,n'}^2 - j_{l,n}^2}, \quad n \neq n'. \end{aligned} \quad (13)$$

As we see the coefficients  $g_{\beta'\beta'}$  are antisymmetric. This is a direct consequence of orthonormality relation (5). From expressions (7) it follows that the coefficients  $q_{\alpha\beta}$  in (2) can be chosen diagonal with respect to the quantum numbers  $l$  and  $m$ , and are independent on  $m$ :

$$q_{\beta\beta'} = \delta_{ll'} \delta_{mm'} q_{nn'}^l. \quad (14)$$

Now the expansion for the eigenfunctions takes the form

$$\varphi_{lmn} = \sum_{n'=1}^{\infty} q_{nn'}^l \phi_{lmn'}(x, a(t)), \quad (15)$$

with the following infinite set of coupled differential equations for the coefficients:

$$\ddot{q}_{nn'}^l + \omega_{ln'}^2(t) q_{nn'}^l = 2h \sum_{p=1}^{\infty} \dot{q}_{np}^l a_{pn'}^l + \hbar \sum_{p=1}^{\infty} q_{np}^l a_{pn'}^l + h^2 \sum_{p,s=1}^{\infty} q_{np}^l a_{ps}^l a_{n's}^l. \quad (16)$$

If the sphere is asymptotically static at past and future then the in- and out- vacuum states can be defined by using the solutions for coefficients corresponding to the in- and out-modes  $\varphi_{\alpha}^{(\text{in})}(t)$ ,  $\varphi_{\alpha}^{(\text{out})}(t)$  with asymptotics

$$q_{\alpha\beta}^{(\text{in})}(t) \rightarrow e^{-i\omega_{\alpha}^{\text{in}} t} \delta_{\alpha\beta}, \quad t \rightarrow -\infty \quad (17)$$

$$q_{\alpha\beta}^{(\text{out})}(t) \rightarrow e^{-i\omega_{\alpha}^{\text{out}} t} \delta_{\alpha\beta}, \quad t \rightarrow \infty, \quad (18)$$

where we use the notations

$$\omega_{\alpha}^{\text{in}} = \frac{j_{l,n}}{a_-}, \quad \omega_{\alpha}^{\text{out}} = \frac{j_{l,n}}{a_+}, \quad a_{\pm} = \lim_{t \rightarrow \pm\infty} a(t) \quad (19)$$

for the corresponding eigenfrequencies. The field operator in the Heisenberg representation may be expanded in terms of the corresponding eigenfunctions

$$\varphi_{\alpha}^{(s)} = \sum_{\beta} q_{\alpha\beta}^{(s)}(t) \phi_{\beta}(x, a(t)), \quad s = \text{in, out} \quad (20)$$

as

$$\varphi(x, t) = \sum_{\alpha} [a_{\alpha}^{(s)} \varphi_{\alpha}^{(s)} + a_{\alpha}^{(s)+} \varphi_{\alpha}^{(s)*}]. \quad (21)$$

The in and out vacuum states  $|\text{in}\rangle$ ,  $|\text{out}\rangle$  are defined in accordance with

$$a_{\alpha}^{(s)} |s\rangle = 0. \quad (22)$$

The corresponding eigenfunctions are related by the Bogoliubov transformation

$$\varphi_{\alpha}^{(\text{in})} = \sum_{\beta} (\alpha_{\alpha\beta} \varphi_{\beta}^{(\text{out})} + \beta_{\alpha\beta} \varphi_{\beta}^{(\text{out})*}), \quad (23)$$

with Bogoliubov coefficients  $\alpha_{\alpha\beta}$  and  $\beta_{\alpha\beta}$ . Substituting instantaneous basis expansion, multiplying by  $\varphi_{\gamma}^*$  and integrating over the region inside the sphere we obtain the corresponding relation between expansion coefficients:

$$q_{\alpha\gamma}^{(\text{in})}(t) = \sum_{\beta} (\alpha_{\alpha\beta} q_{\beta\gamma}^{(\text{out})} + \beta_{\alpha\beta} q_{\beta\gamma}^{(\text{out})*}). \quad (24)$$

Here we used the formula

$$\int \phi_{\beta}^* \phi_{\beta'}^* d^3x = \delta_{nn'} \delta_{ll'} \delta_{m, -m'}, \quad (25)$$

and the fact that expansion coefficients are independent on  $m$ . Now by taking into account relation (14) the Bogoliubov transformation can be written in the form

$$q_{nn'}^{(\text{in})l}(t) = \sum_p (\alpha_{np}^l q_{pn'}^{(\text{out})l} + \beta_{np}^l q_{pn'}^{(\text{out})l*}). \quad (26)$$

In particular, taking into account (17), in the limit  $t \rightarrow \infty$  from (24) we receive

$$q_{\alpha\gamma}^{\text{in}}(t) = \alpha_{\alpha\beta} e^{-i\omega_{\beta}^{\text{out}} t} + \beta_{\alpha\beta} e^{i\omega_{\beta}^{\text{out}} t}, \quad t \rightarrow +\infty. \quad (27)$$

The mean number of out particles with quantum number  $\alpha$  in the in-vacuum state is determined by the Bogoliubov coefficient  $\beta_{\sigma\alpha}$  as

$$\langle \text{in} | N_{\alpha} | \text{in} \rangle = \sum_{\sigma} |\beta_{\sigma\alpha}|^2, \quad \beta_{\sigma\alpha} = -(\varphi_{\sigma}^{(\text{in})}, \varphi_{\alpha}^{(\text{out})*}), \quad (28)$$

with the standard Klein-Gordon scalar product (see, for instance, [4]). From (28) it follows that  $\beta_{\alpha'\alpha} = \beta_{n'n}^l \delta_{ll'} \delta_{m, -m'}$  and hence the total number of created scalar particles is given by

$$\langle \text{in} | N | \text{in} \rangle = \sum_{lmn} \langle \text{in} | N_{lmn} | \text{in} \rangle = \sum_{l=0}^{\infty} (2l+1) \sum_{n, n'=1}^{\infty} |\beta_{n'n}^l|^2 \quad (29)$$

The coefficients  $\alpha_{np}^l$  and  $\beta_{np}^l$  in (24) are determined from the solutions of infinite set of coupled differential equations (16) with time dependent coefficients. The problem can be simplified in limiting cases when various approximations can be used (see, for example, [9]-[16] for the case of plane boundaries). In next section we consider the adiabatic approximation.

### 3 Adiabatic approximation

From the form of equations (16) it follows that there are two types of effects which lead to the particle creation (see also [15]). The first one, called squeezing of the vacuum, is due to the nonstationary eigenfrequencies  $\omega_{ln}(t)$  as a result of a dynamical change of the radius of the sphere and is described by the second term on the left of (16). The second one, referred as acceleration effect, is due to the motion of the boundary and comes from the terms on the right of (16). Due to the antisymmetry of the coefficients  $a_{nn'}^l$ , the squeezing- and acceleration- effects give additive contributions to the number of created particles per mode to first non-vanishing order of perturbation theory. In this section we will consider the squeezing contribution to the number of particles. Note that when the sphere is moving on a time scale slow on the scale of the created quanta frequencies (adiabatic approximation)(see [11] and references therein) the terms on the right of (16) containing the derivatives of the sphere radius are small and the squeezing effect is dominant. In the adiabatic approximation the matrix  $q_{\alpha\beta}$  can be chosen in diagonal form  $q_{\alpha\beta} = q_{\alpha} \delta_{\alpha\beta}$  and  $q_{nn'}^l = q_{ln} \delta_{nn'}$  and from (16) we receive

$$\ddot{q}_{ln}(t) + \omega_{ln}^2(t) q_{ln}(t) = 0. \quad (30)$$

As it follows from (16) the necessary conditions for the adiabatic approximation are

$$\dot{a}^2, \quad a\ddot{a} \ll j_{l,n}^2. \quad (31)$$

As an example let us consider an exactly solvable case when

$$a(t) = \frac{1}{\sqrt{A + B \tanh(t/t_0)}}, \quad A > |B|, \quad (32)$$

where  $A, B$  and  $t_0$  are constants. This motion corresponds to the sphere contraction for  $B > 0$  and expansion for  $B < 0$ . The corresponding frequencies are

$$\omega_{ln}(t) = j_{l,n} \sqrt{A + B \tanh(t/t_0)}, \quad (33)$$

$$\omega_{ln}^{\text{in}} = \frac{j_{l,n}}{a(-\infty)} = j_{l,n} \sqrt{A - B}, \quad \omega_{ln}^{\text{out}} = \frac{j_{l,n}}{a(\infty)} = j_{l,n} \sqrt{A + B}. \quad (34)$$

Now we need to solve the equation (30) with  $\omega_{ln}(t)$  given by (33). The corresponding solutions are given by hypergeometric function. The normalized in- and out- modes are given by formula [4]

$$q_{ln}^s(t) = (2\omega_{ln}^s)^{-1/2} \exp[-i\omega_{ln}^+ t - i\omega_{ln}^- t_0 \ln[2 \cosh(t/t_0)]] \times \\ \times {}_2F_1(1 + i\omega_{ln}^- t_0, i\omega_{ln}^- t_0; 1 \mp i\omega_{ln}^s t_0; \frac{1}{2}(1 \pm \tanh(t/t_0))), \quad s = \text{in, out}, \quad (35)$$

where upper/lower sign corresponds to the in/out- modes, and

$$\omega_{ln}^\pm = \frac{1}{2}(\omega_{ln}^{\text{out}} \pm \omega_{ln}^{\text{in}}). \quad (36)$$

Using the diagonality for the coefficients  $q_{nn'}^l$  over  $nn'$  we conclude that the same is the case for Bogoliubov coefficients in (26):

$$\alpha_{nn'}^l = \alpha_{ln} \delta_{nn'}, \quad \beta_{nn'}^l = \beta_{ln} \delta_{nn'}. \quad (37)$$

Now the Bogoliubov transformation (26) can be written as

$$q_{ln}^{(\text{in})} = \alpha_{ln} q_{ln}^{(\text{out})} + \beta_{ln} q_{ln}^{(\text{out})*}. \quad (38)$$

Using the linear relation between hypergeometric functions, similar to [4] for the coefficients in this formula one finds

$$\alpha_{ln} = \left( \frac{\omega_{ln}^{\text{out}}}{\omega_{ln}^{\text{in}}} \right)^{1/2} \frac{\Gamma(1 - i\omega_{ln}^{\text{in}} t_0) \Gamma(-i\omega_{ln}^{\text{out}} t_0)}{\Gamma(-i\omega_{ln}^+ t_0) \Gamma(1 - i\omega_{ln}^+ t_0)}, \quad (39)$$

$$\beta_{ln} = \left( \frac{\omega_{ln}^{\text{out}}}{\omega_{ln}^{\text{in}}} \right)^{1/2} \frac{\Gamma(1 - i\omega_{ln}^{\text{in}} t_0) \Gamma(i\omega_{ln}^{\text{out}} t_0)}{\Gamma(i\omega_{ln}^- t_0) \Gamma(1 + i\omega_{ln}^- t_0)}. \quad (40)$$

The mean number of particles produced through the modulation of the single scalar mode is

$$\langle \text{in} | N_{ln} | \text{in} \rangle = |\beta_{ln}|^2 = \frac{\sinh^2(\pi \omega_{ln}^- t_0)}{\sinh(\pi \omega_{ln}^{\text{in}} t_0) \sinh(\pi \omega_{ln}^{\text{out}} t_0)}. \quad (41)$$

The total number of particles produced is obtained by taking the sum over all the oscillation modes inside the sphere:

$$\langle \ln|N| \ln \rangle = \sum_{l=0}^{\infty} (2l+1) \sum_{n=1}^{\infty} \frac{\sinh^2[\pi j_{l,n} t_0 (\sqrt{A+B} - \sqrt{A-B})/2]}{\sinh(\pi j_{l,n} t_0 \sqrt{A-B}) \sinh(\pi j_{l,n} t_0 \sqrt{A+B})}. \quad (42)$$

Therefore the energy related to the particles production inside the sphere is given by

$$E = \sum_{l=0}^{\infty} \sum_{n=1}^{\infty} N_{ln} \omega_{l,n}^{\text{out}} = \sum_{l=0}^{\infty} \sum_{n=1}^{\infty} (2l+1) \frac{\sinh^2[\pi j_{l,n} t_0 (\sqrt{A+B} - \sqrt{A-B})/2]}{\sinh(\pi j_{l,n} t_0 \sqrt{A-B}) \sinh(\pi j_{l,n} t_0 \sqrt{A+B})} j_{l,n} \sqrt{A+B}. \quad (43)$$

It can be seen that the similar results may be obtained for another type of boundary conditions, for instance, for the Neumann or more general Robin conditions taking instead of  $j_{l,n}$  the corresponding eigenvalues.

As an another example let us consider the case in which the sphere radius varies as (for the corresponding problem in the case of parallel plates see [11])

$$a(t) = a_0 \left[ 1 + \frac{b^2}{\cosh^2(t/t_0)} \right]^{-1/2} \quad (44)$$

with  $a_0, b, t_0$  being constants. This function describes a pulse with characteristic duration  $t_0$  and modulation strength  $b$ . As in the previous example the corresponding in- and out-solutions to equation (30) can be expressed in terms of hypergeometric function (see, for instance, [24]). Expanding in-modes in terms of the out functions for the mean number of scalar particles with a given mode one receives

$$\langle \ln|N_{ln}| \ln \rangle = \frac{\cos^2[(\pi/2) \sqrt{1 + (2j_{l,n} b t_0 / a_0)^2}]}{\sinh^2(\pi j_{l,n} t_0 / a_0)}. \quad (45)$$

An alternative way to obtain this result is to use the corresponding quantum mechanical reflection probability given in [24]. Now we can find the total energy released by adding the energy of each quantum:

$$E = \frac{1}{a_0} \sum_{l=0}^{\infty} (2l+1) \sum_{n=1}^{\infty} j_{l,n} \frac{\cos^2[(\pi/2) \sqrt{1 + (2j_{l,n} b t_0 / a_0)^2}]}{\sinh^2(\pi j_{l,n} t_0 / a_0)}. \quad (46)$$

Recall that we have obtained formulae (41) and (45) neglecting the terms on the right of equations (16), and hence these formulae give the number of particles produced due to the squeezing effect. In general, the acceleration effect will give additional contribution to the particle creation. It is important to stress again that these two types of effects give additive contributions to first non-vanishing order of perturbation theory. As a result the formulae (41) and (45) as giving a part of the quanta number due to the squeezing effect are valid for more general situations, when additional contributions to the total number of particles have to be taken into account due to the acceleration effect.

By using the results given above for a scalar field we can obtain the number of produced particles in the physically more realistic electromagnetic case. For this note that electromagnetic field in a spherical cavity is a superposition of two types of modes: transverse electric (TE) and transverse magnetic (TM). The TE-modes correspond to the scalar

modes with Dirichlet boundary condition (excluding the  $l = 0$  mode) and TM-modes correspond to the scalar modes with mixed type (Robin) boundary condition

$$j_l(\omega a) + \omega a j_l'(\omega a) = 0, \quad (47)$$

on the sphere surface  $r = a$ . Now the number of photons produced can be presented in the form

$$\langle \text{in} | N_{ln} | \text{in} \rangle = \langle \text{in} | N_{ln}^{(\text{TE})} | \text{in} \rangle + \langle \text{in} | N_{ln}^{(\text{TM})} | \text{in} \rangle, \quad l = 1, 2, \dots \quad (48)$$

where the first summand on the right is given by formulae (41) and (45), and the corresponding expressions for the second summand are obtained from these formulae by replacement  $j_{l,n} \rightarrow \gamma_{l,n}$ , where  $\omega a = \gamma_{l,n}$  are solutions to the equation (47).

## 4 Conclusion

In this paper we have considered the particle creation from the scalar vacuum by moving spherical boundary. For the general case the set of equations is derived for the instantaneous basis expansion coefficients and corresponding Bogoliubov coefficients are considered. To solve these equations and to obtain the particle number, various approximations can be used. Here we use the adiabatic approximation, assuming small velocities for the sphere motion or large frequencies for the radiation quanta. Specific examples are considered when the number of particles produced can be explicitly found. The first one corresponds to the expansion or contraction of the sphere between two finite values of the radius, and second one corresponds to the pulse described by (44). The results for the electromagnetic field can be obtained by summing the contributions from TE and TM modes, corresponding to the scalar modes with Dirichlet and special Robin type boundary conditions. In general, when the acceleration effect in the particle creation cannot be neglected the formulae presented in previous section give only the squeezing parts of the total number of particles. To first non-vanishing order of perturbation theory the contributions from these two types of effects are additive.

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